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Dr. Niclas received the outstanding publications award in 1962 of the German Society of Radio Engineers.

Weakly Coupled Dielectric Resonators

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Abstract —**The resonant modes of a pair of coupled resonators of high ϵ_r are considered in the limit of large spacings D between resonators. Attention is focused on the lowest "magnetic-moment" mode, where the coupling effect leads to a split of the original mode into an even and an odd part. Formulas are obtained for the coupling coefficient, the resonant frequencies and Q of the modes. They are strikingly similar to those for weakly-coupled $R-L-C$ circuits. The accuracy of the formulas is verified by comparing their predictions with direct numerical data, available for coupled circular cylindrical resonators.**

I. Introduction

THE RESONANT modes of a dielectric resonator of high ϵ_r are obtained by solving the differential problem

$$-\text{curl curl}\,\bar{h}_m + k_m^2 \bar{h}_m = 0 \qquad \text{in the dielectric}$$
$$\text{curl}\,\bar{h}_m = 0 \qquad \text{outside the dielectric.} \quad (1)$$

The eigenvector $\bar{h}_m$ must be continuous across the boundary surface, and of order $1/R^2$ (or higher) at large distances [1]. The numerical solution of (1), a difficult three-dimensional problem, simplifies when the body is of revolution (e.g., a sphere or a circular cylinder [2]). In the coupled structure shown in Fig. 1, (1) must be solved in the presence of a composite dielectric consisting of volumes **1** and **2**. When the spacing between **1** and **2** is small, the field distribution in each volume will be significantly perturbed with respect to that of the resonator isolated in free space. Any simplifying feature, such as symmetry of revolution, will be lost, except in structures of the type shown in Fig. 4.

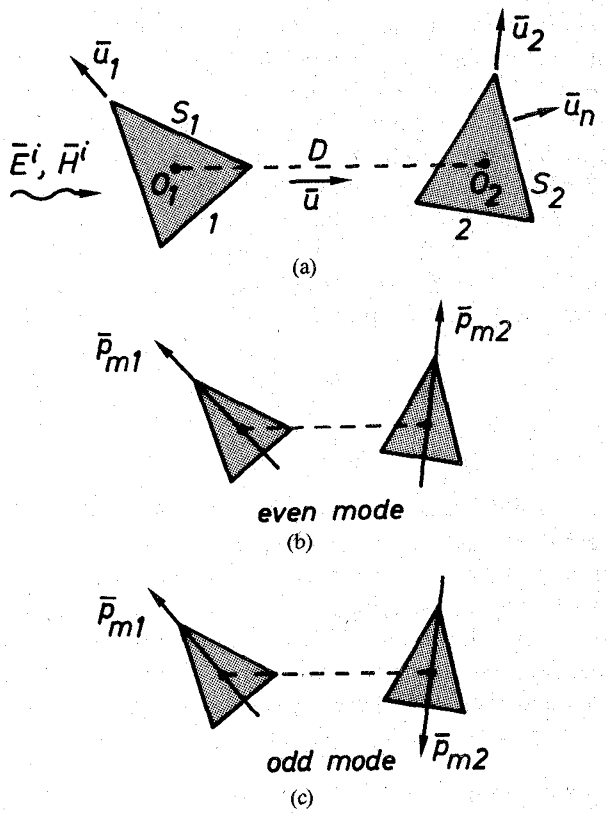

Fig. 1. Coupled resonators with corresponding resonant modes.

The conclusion is clear: the three-dimensional problem must be solved. To avoid this very arduous task, approximations have been made in the past [3], [4]. Cohn, for example, represents dielectric resonators by conducting loops carrying currents I, and endowed with L, C, and

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mutual inductance M. In the present paper, we shall keep the idea of the equivalent circuit, but will seek to obtain its parameters by direct field methods. Given the complexity of the problem, we shall assume that the resonators are "far apart". An asymptotic approach is now possible, which is valid for large spacings D. Truly, resonators in structures such as microwave filters are normally tightly-coupled. It will be interesting, however, to investigate how good (or bad) our asymptotic results are at moderate spacings D. Good results would allow us to extend the applicability of our (very simple) formulas to many realistic configurations.

To simplify the problem, while retaining the main features of the method, we shall assume (1) that the two resonators are identical, although oriented arbitrarily with respect to the line joining their reference points $0_1 0_2$, and (2) that the analysis centers on the lowest resonant mode, which radiates like a magnetic dipole $\bar{p}_m$ [1].

Central to our analysis is the concept of "large D". We shall assume D to be large enough for (1) the exterior field $\bar{h}_m$ of the isolated resonator to be undistinguishable, for all practical purposes, from that of a dipole $\bar{p}_m$ located in 0. From the example of the ring-resonator, for which accurate data are available [2], it can be expected that this assumption will be acceptable as soon as D is larger than twice the largest dimension of the resonator, and (2) the exterior field $\bar{h}_m$ of the isolated resonator to be practically constant over the volume occupied by the other resonator.

On the other hand, we shall restrict D to remain small with respect to λ_0, the wavelength in vacuo. This restriction ensures that resonator **2** lies in the *static-field* of resonator **1** (and conversely). It is useful, in that respect, to note that the angular frequency at resonance, and the corresponding wavelength, are given by

$$\omega_m = \frac{1}{N} k_m c$$

$$(\lambda_m)_0 = \frac{2\pi}{k_m} N. \tag{2}$$

For large indices of refraction $N = \sqrt{\epsilon_r}$ (the limit in which we are interested), λ_0, and therefore the extent of the static region, will be correspondingly large. For a sphere, for example, the lowest value of $2\pi/k_m$ is twice the radius $\boldsymbol{a}$, hence λ_0 is N times $2\boldsymbol{a}$. The theory developed in the sequel shows striking similarities with that of coupled resonant LC circuits. We shall therefore start with a short enumeration of the main results of the latter.

II. Some Results from Coupled LC Circuit Theory

Fig. 2(a) shows two magnetically coupled resonant circuits. The mutual inductance M is positive when the fluxes created by positive I_1 and I_2 have the same sense in the coils, and negative in the opposite case. Let γ be the coupling coefficient M/L, and $\omega_r = (LC)^{-1/2}$ the resonant angular frequency of the isolated circuit. The coupled circuit can resonate in two modes, characterized by different resonant frequencies, viz. (Fig. 2(b)) a) the *even* mode,

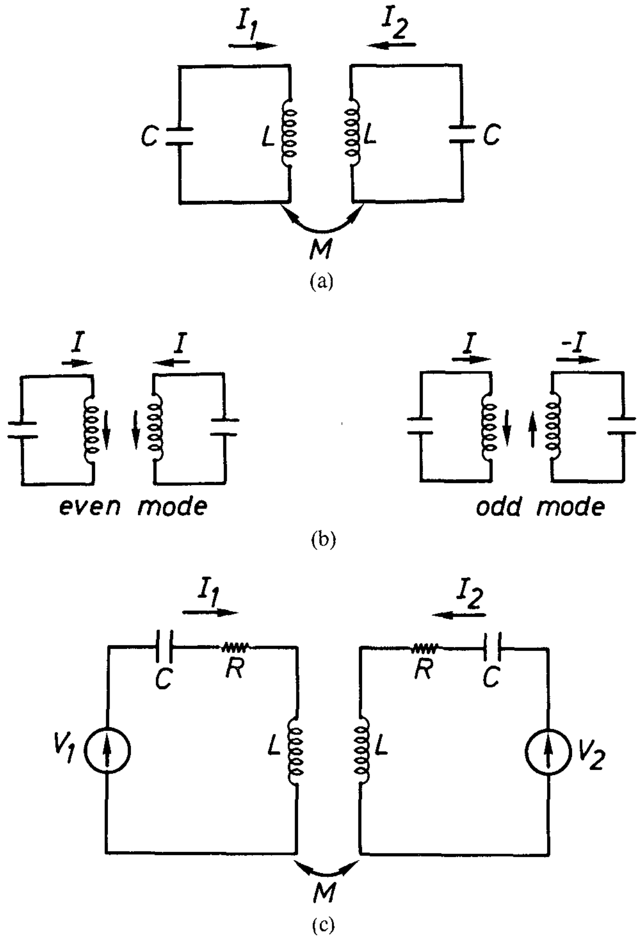

Fig. 2. Coupled resonant circuits with fundamental modes of oscillation.

where

$$\omega_e^2 = \frac{\omega_r^2}{1+\gamma}$$

$$I_1 = I_2 \tag{3}$$

and b) the *odd* mode, where

$$\omega_0^2 = \frac{\omega_r^2}{1-\gamma}$$

$$I_1 = -I_2. \tag{4}$$

When γ is positive the higher frequency is that of the odd mode, the lower one that of the even mode. When γ is negative, the opposite occurs. Notice also that the even or odd character depends on the (arbitrary) choice of the positive directions for I_1 and I_2.

Let us now introduce a resistance R in series with each LC circuit. In the absence of sources, the following relationship exists between I_1 and I_2:

$$I_1 = -\frac{\gamma\omega^2}{\omega^2 - \omega_r^2 - j\dfrac{\omega\omega_r}{Q}} I_2$$

$$\approx -\frac{\gamma\omega^2}{\omega^2 - \omega_r^2\left(1 + \dfrac{j}{Q}\right)} I_2. \tag{5}$$

Here, Q is the quality factor $\omega_r L/R$ (assumed high), and the right-hand expression in (5) is valid in the immediate vicinity of ω_r. When the coupled structure oscillates freely, the even and odd modes have different quality factors, given respectively by

$$Q_e = \frac{\omega_e L(1+\gamma)}{R} = \frac{\omega_r L}{R}\sqrt{1+\gamma} = Q\sqrt{1+\gamma}$$

$$Q_0 = \frac{\omega_0 L(1-\gamma)}{R} = \frac{\omega_r L}{R}\sqrt{1-\gamma} = Q\sqrt{1-\gamma}. \quad (6)$$

Under forced oscillations (Fig. 2(c)), the circuit equations become

$$\frac{1}{L}V_1 = j\left(\omega - \frac{\omega_r^2}{\omega} - \frac{j\omega_r}{Q}\right)I_1 + j\omega\gamma I_2$$

$$\frac{1}{L}V_2 = j\omega\gamma I_1 + j\left(\omega - \frac{\omega_r^2}{\omega} - j\frac{\omega_r}{Q}\right)I_2. \quad (7)$$

III. Free Oscillations of Coupled Dielectric Resonators

Following the policy outlined in the Introduction, we shall assume that each resonator in the structure of Fig. 1(a) behaves like an isolated resonator immersed in the (uniform) weak field $\bar{H}$ of the other. Let us first consider resonator **2**. The incident field in **2**, denoted by $\bar{H}_2$, excites the various modes of **2**. In particular, a mode of the magnetic dipole type will give a contribution [5]

$$\bar{H} = -\frac{k^2}{k^2 - k_m^2}\frac{\bar{p}_{m2}\cdot\bar{H}_2}{N_m^2}\bar{h}_{m2}(r). \quad (8)$$

In (8), N_m^2 is a normalization constant, given by

$$N_m^2 = \iiint_{\substack{\text{all}\\\text{space}}} |\bar{h}_m|^2\, dV. \quad (9)$$

The symbol k^2 denotes the wavenumber in the dielectric, viz.

$$k^2 = \omega^2\epsilon\mu_0 - j\omega\mu_0\sigma$$
$$= \omega^2\epsilon_0\mu_0 N^2 - j\omega\mu_0\sigma = \omega^2\epsilon\mu_0\left(1 - \frac{j}{Q_d}\right). \quad (10)$$

Here, Q_d is the quality factor of the dielectric material. The vector $\bar{p}_m$ is the eigen-magnetic moment of the resonant mode, given by

$$\bar{p}_m = \frac{1}{2}\iiint_{\text{diel}} \bar{r}\times\bar{J}\,dV = \frac{1}{2}\iiint_{\text{diel}} \bar{r}\times \operatorname{curl}\bar{h}_m\, dV. \quad (11)$$

This moment has a well-defined direction, characterized by unit vectors $\bar{u}_1$ and $\bar{u}_2$ in Fig. 1(a). We write $\bar{p}_{m1} = p_m\bar{u}_1$ and $\bar{p}_{m2} = p_m\bar{u}_2$. In the vicinity of a resonant frequency, if Q_d is very high, the amplitude coefficient of the resonant mode becomes very large, and the magnetic field $\bar{H}$ produced by **2** takes the approximate form

$$\bar{H} \approx F_2\bar{h}_{m2} = -\frac{\omega^2}{\omega^2 - \omega_m^2\left(1 + \frac{j}{Q}\right)}\frac{\bar{u}_2\cdot\bar{H}_2}{N_m^2}p_m\bar{h}_{m2}(\bar{r}). \quad (12)$$

In this formula, we have introduced the total quality factor Q, given by

$$\frac{1}{Q} = \frac{1}{Q_d} + \frac{1}{Q_r} \quad (13)$$

where Q_r is the radiation quality factor. This quantity will be discussed in Section IV.

At large distances, the field (12) is the field of a magnetic dipole of moment

$$\bar{P}_{m2} = F_2\bar{p}_{m2} = -\frac{\omega^2}{\omega^2 - \omega_m^2\left(1 + \frac{j}{Q}\right)}\frac{\bar{u}_2\cdot\bar{H}_2}{N_m^2}p_m^2\bar{u}_2. \quad (14)$$

A similar equation can be written for $\bar{P}_{m1}$, the magnetic moment of resonator **1**. The latter moment produces the field $\bar{H}_2$ in which resonator **2** is immersed. From classical magnetostatics, $\bar{H}_2$ is (Fig. 1(a))

$$\bar{H}_2 = \frac{1}{2\pi}\frac{P_{m1}}{D^3}(\bar{u}\cdot\bar{u}_1)\bar{u} + \frac{1}{4\pi}\frac{P_{m1}}{D^3}\bar{u}\times(\bar{u}\times\bar{u}_1). \quad (15)$$

Inserting (15) in (14) yields, for the mode excitation coefficient F_2, the value

$$F_2 = -\frac{\omega^2}{\omega^2 - \omega_m^2\left(1 + \frac{j}{Q}\right)}\frac{p_m^2}{N_m^2}\frac{1}{4\pi D^3}$$
$$\cdot\left[3(\bar{u}\cdot\bar{u}_1)(\bar{u}\cdot\bar{u}_2) - (\bar{u}_1\cdot\bar{u}_2)\right]F_1. \quad (16)$$

Comparison with (5) shows that there is a complete parallelism between the responses of the $R-L-C$ circuits and of the dielectric resonators. It is seen, indeed, that the I's and the F's satisfy analogous equations. The coupling coefficient for the dielectric bodies is clearly

$$\gamma = \frac{1}{2\pi D^3}\frac{p_m^2}{N_m^2}\left[1.5(\bar{u}\cdot\bar{u}_1)(\bar{u}\cdot\bar{u}_2) - 0.5(\bar{u}_1\cdot\bar{u}_2)\right]. \quad (17)$$

The factor $p_m^2/2\pi N_m^2$ has the dimension of a volume. It can therefore be written as

$$\frac{1}{2\pi}\frac{p_m^2}{N_m^2} = L^3 \quad (18)$$

where L is a characteristic length of the resonator. For a sphere of radius a, for example, it is 0.85 a. The coupling coefficient now becomes

$$\gamma = \left(\frac{L}{D}\right)^3 0 \quad (19)$$

where 0, the term between brackets in (17), is an orientation factor, the value of which lies between $+1$ and -1. It is seen that γ decreases proportionally with $1/D^3$.

The coupled resonator structure can oscillate in two modes

a) the *even* mode, with frequency given by (3), $F_1 = F_2$, and moments $\bar{P}_{m1}$ and $\bar{P}_{m2}$ oriented as in Fig. 1(b), and
b) the *odd* mode, with frequency given by (4), $F_1 = -F_2$, and moments $\bar{P}_{m1}$ and $\bar{P}_{m2}$ oriented as in Fig. 1(c).

Let us consider the important particular case of a resonant structure with symmetry plane π. For such case:

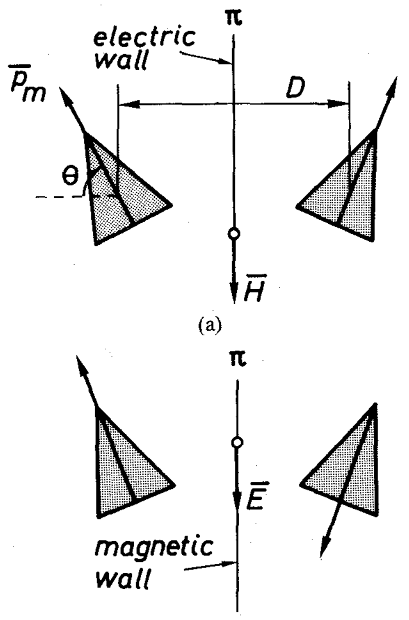

Fig. 3. Coupled resonators located symmetrically with respect to a plane π. (a) Even mode. (b) Odd mode.

a) An *electric wall* condition exists in π for the *even* mode (Fig. 3(a)). This mode, therefore, describes the field of the isolated dielectric resonator in the presence of a metallic ground plane. The relevant coupling coefficient, valid for sufficiently large D, is

$$\gamma = -\frac{1}{2}\left(\frac{L}{D}\right)^3 (1+\cos^2\theta). \tag{20}$$

It is seen that γ is negative, and lies between $-L^3/2D^3$ ($\overline{P}_m$ parallel with the conducting plane) and $-L^3/D^3$ ($\overline{P}_m$ perpendicular to that plane). As γ is negative, the presence of the wall *increases* the resonant frequency.

b) A *magnetic wall* boundary condition exists in π for the *odd* mode (Fig. 3(b)). The presence of the wall now *decreases* the resonant frequency.

IV. Forced Oscillations of Coupled Dielectric Resonators. The Radiation Q

A. The Amplitude of the Forced Oscillations

Let the coupled structure of Fig. 1(a) be immersed in an incident field $\overline{E}^i$, $\overline{H}^i$, of frequency close to ω_m. The coefficients of excitation of the modes, F_1 and F_2, are still given by (12), but the field $\overline{H}_2$ now consists of the incident value $\overline{H}_2^i$ plus the field produced by resonator 1. We write, from (15),

$$\overline{H}_2 = \overline{H}_2^i + \frac{1}{2\pi}\frac{P_{m1}}{D^3}(\bar{u}\cdot\bar{u}_1)\bar{u} + \frac{1}{4\pi}\frac{P_{m1}}{D^3}\bar{u}\times(\bar{u}\times\bar{u}_1). \tag{21}$$

When this value is inserted in the formula for F_2, it yields

$$F_2 = -\frac{\omega^2}{\omega^2-\omega_m^2\left(1+\frac{j}{Q}\right)}\left[\frac{p_m}{N_m^2}(\bar{u}_2\cdot\overline{H}_2^i)+\gamma F_1\right]. \tag{22}$$

This relationship can be rewritten as

$$-j\omega\frac{p_m}{N_m^2}(\bar{u}_2\cdot\overline{H}_2^i) = j\omega\gamma F_1 + j\left[\omega-\frac{\omega_m^2}{\omega}\left(1+\frac{j}{Q}\right)\right]F_2. \tag{23}$$

In this form it is clearly similar to (7), the corresponding equation for the coupled $R-L-C$ circuits. More specifically, the left-hand member of (23) plays the role of the forcing function V_2/L, and the right-hand member reproduces that of (7) provided we remember that, in the vicinity of resonance,

$$\frac{\omega_m^2}{\omega}\frac{1}{Q} = \frac{\omega_m^2}{(\omega_m+\Delta\omega)}\frac{1}{Q} \approx \frac{\omega_m}{Q}. \tag{24}$$

The forced oscillations of the coupled dielectric resonators therefore obey the same laws as those of the coupled $R-L-C$ systems. In particular, the amplitude of oscillation at resonance will be limited by the value of the Q-factor. The latter should therefore be investigated in suitable detail.

B. Reactive Energy

To evaluate Q_r we shall apply the classical formula

$$Q = \frac{\omega \times \text{average reactive energy}}{\text{average dissipated power}}. \tag{25}$$

The reactive energy in that formula is that of the lossless structure. It consists of a magnetic part and an electric part. Detailed calculations show [6] that the time-averaged magnetic energy of the coupled system can be written as

$$\epsilon_m = 2(1\pm\gamma)\epsilon_{mi} \tag{26}$$

where the upper and lower signs correspond, respectively, to the even and odd modes, and where ϵ_{mi} is the magnetic energy of the isolated resonator, assumed excited with coefficient $|F| = |F_1| = |F_2|$. A similar relationship exists for the electric energy, viz.

$$\epsilon_e = 2(1\pm\gamma)\epsilon_{ei} \tag{27}$$

For the isolated resonator $\epsilon_{ei} = \epsilon_{mi}$. It is seen that equipartition of energy still holds at the resonant frequencies of the even and odd modes. Adding (26) and (27) shows that the total reactive energy is given by

$$\epsilon = 2(1\pm\gamma)\epsilon_i. \tag{28}$$

For the configuration of Fig. 3(a), where a single resonator is located in the vicinity of a metallic screen, the reactive energy for the half-space is

$$\epsilon = (1+\gamma)\epsilon_i = \left[1-\frac{1}{2}\left(\frac{L}{D}\right)^3(1+\cos^2\theta)\right]\epsilon_i \tag{29}$$

C. Radiated Power

The power radiated by the coupled resonators is that associated with the total magnetic dipole $\overline{P}_{m1}+\overline{P}_{m2}$. From (14), the total moment is

$$\overline{P}_m = F_1 p_m(\bar{u}_1+\bar{u}_2) \tag{30}$$

for the even mode, and

$$\bar{P}_m = F_1 p_m(\bar{u}_1 - \bar{u}_2) \tag{31}$$

for the odd mode. The justification for adding the magnetic moments rests on the fact that the electric dipole moment $(1/j\omega)\int \bar{J}\,dV$ of the resonator vanishes [1]. This makes $\bar{P}_m$ invariant with respect to a shift of origin, as shown by the elementary calculation

$$\bar{P}_m = \frac{1}{2}\iiint \bar{r} \times \bar{J}\,dV$$
$$= \frac{1}{2}\iiint (\bar{a} + \bar{r}') \times \bar{J}\,dV = \frac{1}{2}\iiint \bar{r}' \times \bar{J}\,dV. \tag{32}$$

The average power radiated by a magnetic dipole is $(1/12)\pi\omega^4 c^{-3}\mu_0|\bar{P}_m|^2$. It follows that the power radiated by the coupled system at resonance is given, in the even mode, by

$$\mathcal{P} = \frac{1}{12\pi}\omega_m^4 c^{-3}\mu_0|F|^2 p_m^2 \frac{2(1+\bar{u}_1\cdot\bar{u}_2)}{(1+\gamma)^2} = \mathcal{P}_i \frac{2(1+\bar{u}_1\cdot\bar{u}_2)}{(1+\gamma)^2} \tag{33}$$

where $\mathcal{P}_i$ is the power radiated (at the resonance frequency ω_m) by the isolated resonator of excitation level $|F|$. In the odd mode

$$\mathcal{P} = \mathcal{P}_i \frac{2(1-\bar{u}_i\cdot\bar{u}_2)}{(1-\gamma)^2}. \tag{34}$$

Combining (28), (33), and (34) gives

$$Q_r = Q_i \frac{(1\pm\gamma)^{5/2}}{1\pm\bar{u}_1\cdot\bar{u}_2}. \tag{35}$$

For the single resonator in front of a metallic screen

$$Q_r = Q_i \frac{\left[1 - \frac{1}{2}\left(\frac{L}{D}\right)^3 (1+\cos^2\theta)\right]^{5/2}}{2\sin^2\theta}. \tag{36}$$

In this expression, Q_i is the quality factor of the isolated resonator, given by [1]

$$Q_i = 3\frac{N^3}{(k_m L)^3}. \tag{37}$$

The previous formulas break down when the total $\bar{P}_m$ vanishes. This happens, for the even mode, when $\bar{u}_1 = -\bar{u}_2$, and for the odd mode, when $\bar{u}_1 = \bar{u}_2$. In both cases, a multipole analysis must be performed to evaluate the radiated power [7]. We shall not solve this problem in its most general form, but only for the particular case discussed in Section V.

V. An Illustrative Example: Coupled Circular Cylinders

A. The Isolated Resonator

The coupled resonator structure shown in Fig. 4(a) possesses symmetry of revolution. It follows that the characteristics of its lowest magnetic-moment mode can be determined by solving a scalar problem. The numerical solution of this problem, for arbitrary D, is available elsewhere [8]. We shall compare these results with the predictions of our asymptotic theory. In the lowest $\bar{p}_m$ mode, the electric field is azimuthal with circular lines of force, and the magnetic field lies in the meridian plane. The choice of positive directions for $\bar{u}_1$ and $\bar{u}_2$ is not evident here, because of the special symmetry of the resonators. With the choice $\bar{u}_1 = -\bar{u}_2 = \bar{u}_z$ shown in Fig. 4, we respect our previous conventions, which associate an electric wall with an even mode, and a magnetic wall with an odd mode. We could also have taken $\bar{u}_1 = \bar{u}_2 = \bar{u}_z$, which would have associated an electric wall with an odd mode, and a magnetic wall with an odd mode. The latter convention is often preferred in microwave applications. The physical results are obviously independent of our choice, which is a matter of taste.

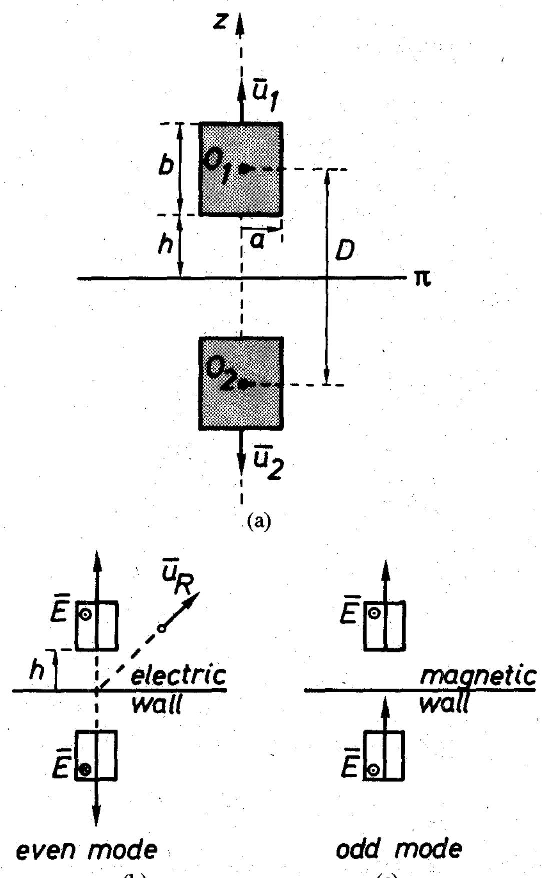

Fig. 4. Coupled circular cylindrical resonators located symmetrically with respect to π.

The characteristics of the isolated resonator have been given elsewhere [2]. Some of them are reproduced in Table I.

B. The Even Mode

The resonant configuration is shown in Fig. 4(b). Values for the coupling coefficient, given by $\gamma = -(L/D)^3$, are given in Table II. It is seen that the coupling coefficient is down to only a few percent for $D/a = 3$. A comparison between the asymptotic values of $k_m a$, as given by (3), and the values obtained by direct numerical evolution is inter-

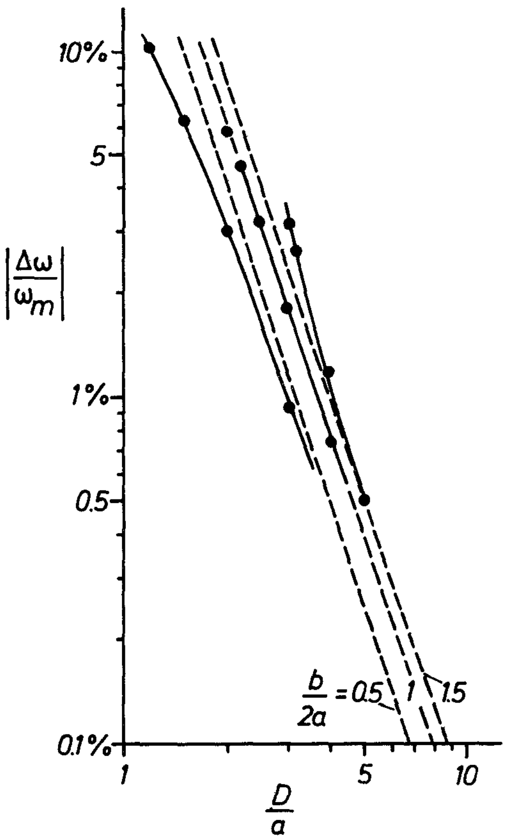

Fig. 5. Shift in the resonance frequency produced by coupling. Data for the even mode with upward frequency shift.

TABLE I
DATA FOR THE ISOLATED RESONATOR

	$\frac{b}{2a}=0.5$	$\frac{b}{2a}=1$	$\frac{b}{2a}=1.5$
$k_m a$	3.2658	2.8276	2.6728
$\frac{Q_i}{N^3}$	0.142	0.147	0.131
$\frac{L}{a}$	0.846	0.966	1.062

TABLE II
COUPLING COEFFICIENT

	$\frac{b}{2a}=0.5$		$\frac{b}{2a}=1$		$\frac{b}{2a}=1.5$	
$\frac{h}{a}=0$	$\frac{D}{a}=1$	$\gamma=-0.605$	$\frac{D}{a}=2$	$\gamma=-0.113$	$\frac{D}{a}=3$	$\gamma=-0.0445$
0.1	1.2	-0.350	2.2	-0.0849	3.2	-0.0367
0.25	1.5	-0.179	2.5	-0.0579	3.5	-0.0280
0.5	2	-0.0757	3	-0.0335	4	-0.0188
1	3	-0.0224	4	-0.0141	5	-0.00961

esting. For $h/a=1$, for example, the asymptotic values for $h/a=1$ and $b/2a$ equal to 0.5, 1, and 1.5, are 3.303, 2.848, and 2.686, respectively. The corresponding numerical values are 3.266, 2.828, and 2.673. The agreement is satisfactory. For larger values of h/a, it is recommended to switch to the asymptotic formula, as it is difficult to obtain numerical results which give comparable accuracy. The agreement is further illustrated in Fig. 5, which shows values of the relative frequency shift $\Delta\omega/\omega_m=\omega_e-\omega_m/\omega_m$. The dots represent the numerical values, the dashed lines the asymptotic values. The convergence of the two sets of curves for increasing D/a is evident.

The calculation of the radiation Q requires evaluation of the radiated power. This point implies knowledge of the far-field. In the present case, the source (i.e., the coupled resonator structure) is small with respect to λ, hence a multipole expansion should be introduced [7]. With the field and current polarities shown in Fig. 4(b), the far-field turns out to be produced by a magnetic quadrupole moment $\bar{\bar{Q}}_m$, and to have the value

$$\bar{E}=-\frac{1}{8\pi}jk_0^3R_c\bar{u}_R\times\left(\bar{u}_R\cdot\bar{\bar{Q}}_m\right)\frac{e^{-jkR}}{R}. \tag{38}$$

A few elementary calculations show that

$$\begin{aligned}\bar{\bar{Q}}_m&=\frac{1}{3}\iiint_{1+2}[\bar{r}\times\bar{J}\bar{r}\,\bar{r}\bar{r}\times\bar{J}]\,dV\\&=\frac{2}{3}DP_{m1}(2\bar{u}_z\bar{u}_z-\bar{u}_x\bar{u}_x-\bar{u}_y\bar{u}_y)\end{aligned} \tag{39}$$

where P_{m1} is the magnetic dipole moment of the isolated dielectric resonator **1**. The free-space radiated power follows as

$$\mathcal{P}=\frac{R_c}{60\pi}k_0^6D^2P_{m1}^2. \tag{40}$$

The resulting Q_{rad} is, from (28), (33), and (37)

$$Q_e=\frac{\omega_e2(1+\gamma)\epsilon_i}{\mathcal{P}}=\frac{10N^2(1+\gamma)^{3\,5}}{(k_mD)^2}Q_i. \tag{41}$$

As Q_i is proportional with N^3, Q_e will be proportional with N^5. Table III shows values calculated from (41), using the data of Tables I and II. They are shown in the column "approx.", while the column "numer." refers to the values obtained by the full numerical solution of the problem [8]. It is seen that the asymptotic formula gives an accuracy of the order of 1 percent as soon as the spacing $2h$ is equal to the diameter $2a$.

C. The Odd Mode

In the odd mode, tighter coupling lowers the resonant frequency. In Fig. 6 we have plotted the absolute value $|\Delta\omega/\omega_m|$ of the frequency shift, using the same conventions as in Fig. 5. The quality factor is given, in the weak coupling approximation, by

$$Q_0=\frac{Q_i}{2}(1-\gamma)^{5/2}. \tag{42}$$

The asymptotic value, for large spacings, is $\frac{Q_i}{2}$. In Table IV, the ratio of Q_0 to $(Q_i/2)$ is given for a few values of the parameters. The column "approx." refers to asymptotic formula (42), the column "numer." to results obtained by full numerical solution of the problem [8]. Both results agree to within a few percent as soon as h/a exceeds one.

D. Additional Verification

The asymptotic formulas have also been tested versus numerical results obtained for coupled spherical resonators

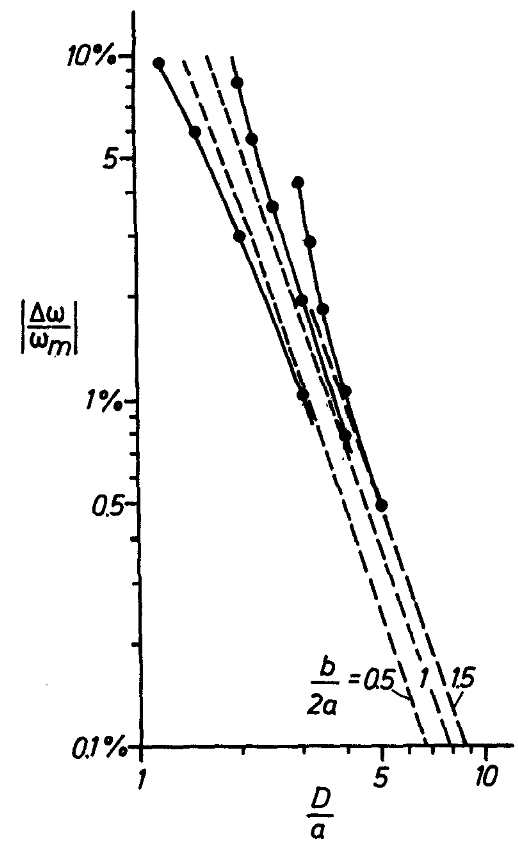

Fig. 6. Shift in the resonance frequency produced by coupling. Data for the odd mode with downward frequency shift.

TABLE III
VALUES OF $1/N^5\, Q_e$

	$\frac{b}{2a}$ = 0.5 approx.	$\frac{b}{2a}$ = 0.5 numer.	$\frac{b}{2a}$ = 1 approx.	$\frac{b}{2a}$ = 1 numer.	$\frac{b}{2a}$ = 1.5 approx.	$\frac{b}{2a}$ = 1.5 numer.
$\frac{h}{a}$ = 0	0.0052	0.0434	0.0302	0.0265	0.0174	0.0142
0.1	0.0205	0.0421	0.0278	0.0248	0.0157	0.0134
0.25	0.0297	0.0373	0.0239	0.0221	0.0136	0.0122
0.5	0.0253	0.0269	0.0181	0.0176	0.0107	0.0102
1	0.0137	0.0139	0.0109	0.0109	0.00709	0.00701

TABLE IV
RATIO OF Q_0 TO $Q_i/2$

	$\frac{b}{a}$ = 0.5 approx.	$\frac{b}{a}$ = 0.5 numer.	$\frac{b}{a}$ = 1 approx.	$\frac{b}{a}$ = 1 numer.	$\frac{b}{a}$ = 1.5 approx.	$\frac{b}{a}$ = 1.5 numer.
$\frac{h}{a}$ = 0	3.33	2.075	1.307	1.568	1.115	1.375
0.1	2.150	1.638	1.226	1.338	1.094	1.213
0.25	1.527	1.340	1.151	1.188	1.071	1.118
0.5	1.205	1.153	1.086	1.094	1.048	1.061
1	1.058	1.052	1.036	1.037	1.024	1.025

[9]. The asymptotic form of the resonant frequency is here

$$ka \approx \pi\left[1 \pm \frac{3}{\pi^2}\left(\frac{a}{D}\right)^3 + \cdots\right]. \tag{43}$$

For a center-to-center distance D of $4a$, i.e., twice the diameter, asymptotic and numerical values turn out to agree within 0.5 percent.

VI. CONCLUSION

In the previous paragraphs, formulas have been derived which give the resonant frequencies and Q of the "magnetic moment" mode of coupled resonators of high ϵ_r. The coupling mechanism splits the resonant mode of the isolated resonator in two separate modes, of even and odd parity, respectively. The formulas, valid for large center-to-center spacings D between resonators, turn out to be identical with those for weakly coupled $R-L-C$ circuits. The coupling coefficient for the dielectric resonator pair is given by

$$\gamma = \frac{L^3}{D^3} 0$$

where L is a characteristic length of the resonator, of the order of its general dimensions, and 0 an orientation factor, comprised between -1 and $+1$. At large distances, γ is small, and the difference between the resonant frequencies ω_o and ω_e of the odd and even modes is correspondingly small. For such cases, the asymptotic formulas of the text are particularly suitable, as a direct numerical procedure requires considerable accuracy on ω_e and ω_o separately to give an accuracy on the relative difference

$$\frac{\omega_e - \omega_o}{\frac{1}{2}(\omega_e + \omega_o)}$$

comparable with that of the asymptotic formula. These remarks have been confirmed by looking at a solved numerical example, that of two coupled circular cylinders with a common axis of revolution. If we extrapolate the results obtained for this configuration, we arrive at the conclusion that asymptotic and direct numerical results agree to within about 1 percent, both in Q and relative frequency difference, as soon as the spacing D between centers is of the order of 1.50 times the maximum dimension $L_{\max}$ of the single resonator. When the spacing is of the order of $1.2L_{\max}$, the coupling coefficient is of the order of 5 percent, and so is the accuracy. It is seen that the asymptotic solution is capable of yielding reasonably accurate values at fairly tight couplings, hence, that it is relevant for the design of structures such as microwave filters.

In many practical applications, the dielectric resonators are located close to a metallic boundary (e.g., the walls of a waveguide), and the assumption of free-space coupling becomes unrealistic. The extension of our theory to this new situation will be discussed in a future article.

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Improved Waveguide Diode Mount Circuit Model Using Post Equivalence Factor Analysis

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***Abstract*—This paper presents an improved wide-band equivalent circuit for a diode mount consisting of a gapped cylindrical post in a rectangular waveguide. The empirical round post to flat strip equivalence factor used in an earlier study by Eisenhart and Khan is replaced by one which is calculated via an accurate analysis. Results indicating the dependence of this equivalence factor on post diameter, post position, and frequency are shown, allowing a more accurate interpretation from the Eisenhart and Khan analysis.**

I. Introduction

This paper is concerned with an analytical determination of the impedance of a diode mount consisting of a cylindrical post in shunt across a rectangular waveguide. The impedance is found at both the diode package terminals on the cylindrical post and the waveguide terminal plane. Specifically, this paper substitutes a theoretical analysis to determine a factor previously approximated through measurement.

The general modeling problem of a diode mount in waveguide has been under study for many years resulting in a large number of papers on the subject. Eisenhart and Khan [1] carried out an extensive analysis, using a dyadic Green's function approach with an extension of the induced EMF method, to obtain expressions for the required impedances. The approach of Eisenhart and Khan was later applied to a two-post mount structure by El-Sayed [2], to a single-post two-gap configuration by Joshi and Cornick [3], to a waveguide diode mount having a coaxial entry by Eisenhart [4], and to a coaxial-line-waveguide junction by Eisenhart *et al.* [5]. Ogiso and Taketomi [6] used this approach to analyze iris-loaded waveguide diode mounts, while Blocker *et al.* [7] applied it to a study of the influence

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